

# Protecting entangled states of two ions by engineering reservoir

Dong Xue, Jian Zou, Lin-Guang Yang, Jun-Gang Li and Bin Shao

Department of Physics, School of Science, and Key Laboratory of Cluster Science of Ministry of Education, Beijing Institute of Technology, Beijing 100081, People's Republic of China

E-mail: zoujian@bit.edu.cn

## Abstract.

We present a proposal for realizing local decoherence-free evolution of given entangled states of two two-level (TL) ions. For two TL ions coupled to a single heavily damped cavity, we can use engineering reservoir scheme to obtain a decoherence-free subspace which can be nonadiabatically controlled by the system and reservoir parameters. Then the local decoherence-free evolution of the entangled states are achieved. And we also discuss the relation between the geometric phases and the entanglement of the two ions under the nonadiabatic coherent evolution.

PACS numbers: 03.67.Pp; 03.67.Mn; 32.80.-t

## 1. Introduction

It is well known that one of the major obstacles for practical manipulation of quantum information is the effect of decoherence due to coupling to environment. So how to protect quantum information from these effects become a major challenge. A number of strategies have been proposed to fight decoherence, such as quantum error correction [1, 2, 3], decoherence-free subspace (DFS) [4, 5, 6, 7], dynamical decoupling (DD) [8, 9, 10], engineering reservoir [11], *etc.* The general framework for DFS was introduced by Zanardi *et al.* [4] in a spin-boson model undergoing both dephasing and dissipation. In the presence of environment, DFS is a set of all states which are immune to decoherence processes induced by interaction with bath. These are groups of states that have robust symmetry properties. The quantum information can be encoded in DFS in order to increase reliability of quantum information processing [12, 13, 14]. On the other hand Carvalho *et al.* have presented a proposal for protecting states of a trapped ion against decoherence, based on engineering of pointer states [15]. By controlling the reservoir, they have applied an indirect control on the protected states, and have shown how to protect squeezed states, approximate phase eigenstates, and superpositions of coherent states. Recently, Prado *et al.* have shown how to protect a nonstationary superposition states of a two-level (TL) ion. Working with an ion trapped inside a dissipative cavity they have constructed two classes of decoherence-free evolution by broadening engineering reservoir scheme [16]. Under the assumption of a squeezed engineered reservoir, Carollo *et al.* have proposed a way to observe the adiabatic geometric phase acquired by a protected state evolving coherently through the adiabatic manipulation of the squeeze parameters of the engineered reservoir [17]. Prado *et al.* have generalized the method proposed by Carollo *et al.* [17] and shown how to observe the geometric phases acquired by the protected nonstationary states even under the nonadiabatic evolution [16]. Authors in Refs. [15,16] only considered one particle system, then a natural question arises, how about multi-particle system? Can we use this engineering reservoir scheme to protect entanglement? In this paper we will answer these questions.

In this paper we introduce a scheme, which generalizes the engineering reservoir scheme presented for a single ion in Ref. [16], for protecting given entangled states of two TL ions by building a class of periodic nonadiabatic coherent evolution. We consider a system consisting of two TL ions coupled resonantly to a dissipative cavity and simultaneously driven by classical fields. For this system, we can obtain a time-independent master equation by the techniques of engineering reservoir. Then we can show that the given initial entangled state can be projected into a DFS, and in particular, modifications of the parameters of the reservoir may result in a controlled time evolution of the protected subspace as a whole. In this way the protected entangled state lying in this subspace evolves periodically. Finally, we calculate the geometric phase of the whole system and the subsystems under the nonadiabatic coherent evolution.

The paper is organized as follows. In Sec. 2, we introduce the model and deduce the

time-independent master equation by engineering reservoir. Then we obtain the DFS and show how to protect given entangled states of two TL ions. In Sec. 3, we calculate the geometric phase of the whole system and the subsystems under the nonadiabatic evolution and give the relation between the geometric phases and the entanglement of the two TL ions. The conclusions are given in Sec. 4.

## 2. Model and results

We consider a system consisting of a pair of two identical TL ions  $A$  and  $B$  coupled resonantly to a single dissipative cavity mode of frequency  $\omega$  with equal coupling strength  $g$ . In this paper we suppose that there are no direct interactions between the two ions. The  $i$ th ion is driven by two classical fields of frequencies  $\omega_l^i$  with coupling strengths  $\Omega_l^i$  ( $i = A, B; l = 1, 2$ ) where ' $i$ ' and ' $l$ ' represent the ion and the classical field respectively. Within the rotating-wave approximation and a sufficiently small Lamb-Dicke parameter (to keep the motional state almost unchanged), the Hamiltonian modeling the system is given by

$$H = \omega a^\dagger a + \frac{\omega_0}{2}(\sigma_z^A + \sigma_z^B) + [(\sum_{l=1,2} \Omega_l^A e^{i(\phi_l^A - \omega_l^A t)} \sigma_{eg}^A + \sum_{l=1,2} \Omega_l^B e^{i(\phi_l^B - \omega_l^B t)} \sigma_{eg}^B + ga(\sigma_{eg}^A + \sigma_{eg}^B)) + \text{H.c.}], \quad (1)$$

where  $a^\dagger$  ( $a$ ) is the creation (annihilation) operator of the cavity mode,  $\omega_0$  is the transition frequency of the ions,  $\phi_l^A$  and  $\phi_l^B$  are the dephasings of the classical fields, and  $\sigma_{kl}^i \equiv |k\rangle_i \langle l|$ ,  $k$  and  $l$  being the ground state  $g$  and excited state  $e$  of the ions. In the interaction picture, the Hamiltonian can be written as:

$$H_1 = [\sum_{l=1,2} (\Omega_l^A e^{i(\phi_l^A - \Delta_l^A t)} \sigma_{eg}^A + \Omega_l^B e^{i(\phi_l^B - \Delta_l^B t)} \sigma_{eg}^B) + \text{H.c.}] + [ge^{-i\Delta t} a(\sigma_{eg}^A + \sigma_{eg}^B) + \text{H.c.}], \quad (2)$$

where  $\Delta = \omega - \omega_0$  and  $\Delta_l^i = \omega_l^i - \omega_0$ . The first and second terms of the Hamiltonian represent the classical field driving and the coupling between the ions and the cavity respectively. We suppose that the cavity mode is heavily damped with decay rate  $\kappa$  and the ions spontaneously decay with rates  $\gamma^i$ . Then the master equation describing the system is given by

$$\dot{\rho} = -i[H_1, \rho] + \kappa D[a]\rho + \sum_i \gamma^i D[\sigma_{ge}^i]\rho, \quad (3)$$

where the Lindblad decoherence superoperator  $D[a]\rho = a\rho a^\dagger - a^\dagger a\rho/2 - \rho a^\dagger a/2$  and  $D[\sigma_{ge}^i]\rho = \sigma_{ge}^i \rho \sigma_{ge}^i - \sigma_{eg}^i \sigma_{ge}^i \rho/2 - \rho \sigma_{eg}^i \sigma_{ge}^i/2$  describe cavity and ionic decays respectively.

Next we engineer the appropriate interaction between the ionic levels and the cavity mode [16]. Concretely, we adjust the classical field  $\omega_1^i$  in resonance with the ionic transition frequency  $\omega_0$ , i.e.,  $\Delta_1^i = 0$ , and the detuning of the field  $\omega_2^i$  to  $\Delta_2^i = -2\Omega_1^i$ . By applying the unitary transformation  $R(t) = R^A(t) \otimes R^B(t)$  to Hamiltonian (2), where  $R^i(t) = \exp[-i\Omega_1^i t(e^{i\phi_1^i} \sigma_{eg}^i + \text{H.c.})] \exp\{-i\Omega_2^i t[\cos(\varphi^i) \sigma_z^i + i\sin(\varphi^i)(e^{-i\phi_1^i} \sigma_{ge}^i - e^{i\phi_1^i} \sigma_{eg}^i)]\}$  with

$\varphi^i = \phi_1^i - \phi_2^i$ , and adjusting  $\Omega_1^i = -\frac{1}{2}\Delta_2^i \gg \Omega_2^i = -\Delta \gg g$ , we can obtain the effective Hamiltonian in the rotating-wave approximation

$$H_2 = \frac{g}{2}[a^\dagger(\sigma_{+-}^A + \sigma_{+-}^B) + a(\sigma_{-+}^A + \sigma_{-+}^B)], \quad (4)$$

where  $|+\rangle_i = \cos(\varphi^i/2)|e\rangle_i + ie^{-i\phi_1^i}\sin(\varphi^i/2)|g\rangle_i$  and  $|-\rangle_i = ie^{i\phi_1^i}\sin(\varphi^i/2)|e\rangle_i + \cos(\varphi^i/2)|g\rangle_i$ .

Now applying the unitary transformation  $R(t)$  to the original master equation (3) we obtain

$$\dot{\rho}' = -i[H_2, \rho'] + \kappa D[a]\rho' + \sum_i \gamma^i D[\sigma'^i] \rho', \quad (5)$$

where  $\rho' = R(t)^\dagger \rho R(t)$  and  $\sigma'^i = R(t)^\dagger \sigma_{ge}^i R(t)$ . In the limit where the cavity decay rate  $\kappa$  is much larger than the other relevant frequencies, the cavity mode can be adiabatically eliminated and we can obtain a time-independent master equation just for the ions [18]

$$\dot{\rho}' = \Gamma D[\sigma_{+-}^A + \sigma_{+-}^B]\rho' + \sum_i \gamma^i D[\sigma'^i] \rho', \quad (6)$$

where  $\Gamma = g^2/\kappa$  is the coupling strength of the engineered reservoir. Furthermore, under the assumption that  $\Gamma$  is much larger than the spontaneous emission rate  $\gamma^i$  of the ions, i.e.,  $\Gamma \gg \gamma^i$ , Eq. (6) becomes

$$\dot{\rho}' = \Gamma D[\sigma_{+-}^A + \sigma_{+-}^B]\rho'. \quad (7)$$

From Eq. (7), the DFS is composed of all the eigenstates of the operator  $\sigma_{+-}^A + \sigma_{+-}^B$  with zero eigenvalue and it is easy to prove that it is spanned by the following orthonormal states: the product state  $|1\rangle \equiv |+\rangle_A |+\rangle_B$  and the maximal entangled state  $|2\rangle \equiv \frac{1}{\sqrt{2}}(|-\rangle_A |+\rangle_B - |+\rangle_A |-\rangle_B)$ . It is easy to see that the state

$$|\Psi_r\rangle = \sqrt{1-r}|1\rangle + \sqrt{r} e^{i\mu}|2\rangle, \quad (1 \geq r \geq 0, \mu \in (0, 2\pi]), \quad (8)$$

or a mixture state of  $|1\rangle$  and  $|2\rangle$  is the equilibrium state of the master equation (7). If the two ions are initially prepared in these states, they would remain the same. However if the ions starts outside this subspace, the situation gets more complicated. Because Eq. (7) is symmetric with respect to exchange of the two ions, one could analyze the system in terms of the antisymmetric

$$|2\rangle = \frac{(|-\rangle_A |+\rangle_B - |+\rangle_A |-\rangle_B)}{\sqrt{2}}$$

and symmetric

$$\begin{aligned} |1\rangle &= |+\rangle_A |+\rangle_B, \\ |3\rangle &= |-\rangle_A |-\rangle_B, \\ |4\rangle &= \frac{(|-\rangle_A |+\rangle_B + |+\rangle_A |-\rangle_B)}{\sqrt{2}}. \end{aligned}$$

subspaces [19]. Generally the initial state can be decomposed into symmetric and antisymmetric components. The symmetric part of the initial state evolves toward the state  $|1\rangle$  and the antisymmetric component  $|2\rangle$  remain the same.

In this paper, for simplicity we suppose that the protected state is pure. Using concurrence as measure of degree of entanglement between two qubits [20], we can obtain that the entanglement degree of the state  $|\Psi_r\rangle$  is equal to  $r$ . Since we obtain the master equation (7) through the unitary transformation  $R(t)$ , the protected equilibrium state  $|\Psi_r\rangle$  is nonstationary in the original interaction picture. Reversing the unitary transformation  $R(t)$ , the state  $|\Psi_r\rangle$  (for simplicity we take  $\Omega_1^A = \Omega_1^B = \Omega_1, \Omega_2^A = \Omega_2^B = \Omega_2$ ), written in the interaction picture is

$$\begin{aligned} |\Psi(t)\rangle &= R(t)|\Psi_r\rangle = c_1(t)|e\rangle_A|e\rangle_B + c_2(t)|e\rangle_A|g\rangle_B \\ &\quad + c_3(t)|g\rangle_A|e\rangle_B + c_4(t)|g\rangle_A|g\rangle_B, \end{aligned} \quad (9)$$

where  $c_1(t)$ ,  $c_2(t)$ ,  $c_3(t)$  and  $c_4(t)$  are time-dependent parameters:

$$\begin{aligned} c_1(t) &= -\frac{ie^{i\mu}\sqrt{r}}{\sqrt{2}}[e^{i\phi_1^B}\cos(\frac{\varphi^A}{2} - \Omega_1 t)\sin(\frac{\varphi^B}{2} - \Omega_1 t) \\ &\quad - e^{i\phi_1^A}\sin(\frac{\varphi^A}{2} - \Omega_1 t)\cos(\frac{\varphi^B}{2} - \Omega_1 t)] \\ &\quad + e^{-i2\Omega_2 t}\sqrt{1-r}\cos(\frac{\varphi^A}{2} - \Omega_1 t)\cos(\frac{\varphi^B}{2} - \Omega_1 t), \\ c_2(t) &= -\frac{e^{i\mu}\sqrt{r}}{\sqrt{2}}[\cos(\frac{\varphi^A}{2} - \Omega_1 t)\cos(\frac{\varphi^B}{2} - \Omega_1 t) \\ &\quad + e^{i(\phi_1^A - \phi_1^B)}\sin(\frac{\varphi^A}{2} - \Omega_1 t)\sin(\frac{\varphi^B}{2} - \Omega_1 t)] \\ &\quad + ie^{-i(\phi_1^B + 2\Omega_2 t)}\sqrt{1-r}\cos(\frac{\varphi^A}{2} - \Omega_1 t)\sin(\frac{\varphi^B}{2} - \Omega_1 t), \\ c_3(t) &= \frac{e^{i\mu}\sqrt{r}}{\sqrt{2}}[\cos(\frac{\varphi^A}{2} - \Omega_1 t)\cos(\frac{\varphi^B}{2} - \Omega_1 t) \\ &\quad + e^{i(\phi_1^B - \phi_1^A)}\sin(\frac{\varphi^A}{2} - \Omega_1 t)\sin(\frac{\varphi^B}{2} - \Omega_1 t)] \\ &\quad + ie^{-i(\phi_1^A + 2\Omega_2 t)}\sqrt{1-r}\sin(\frac{\varphi^A}{2} - \Omega_1 t)\cos(\frac{\varphi^B}{2} - \Omega_1 t), \\ c_4(t) &= -\frac{ie^{i\mu}\sqrt{r}}{\sqrt{2}}[e^{-i\phi_1^A}\sin(\frac{\varphi^A}{2} - \Omega_1 t)\cos(\frac{\varphi^B}{2} - \Omega_1 t) \\ &\quad - e^{-i\phi_1^B}\cos(\frac{\varphi^A}{2} - \Omega_1 t)\sin(\frac{\varphi^B}{2} - \Omega_1 t)] \\ &\quad - e^{-i(\phi_1^A + \phi_1^B + 2\Omega_2 t)}\sqrt{1-r}\sin(\frac{\varphi^A}{2} - \Omega_1 t)\sin(\frac{\varphi^B}{2} - \Omega_1 t). \end{aligned} \quad (10)$$

The  $\Omega_1$ ,  $\Omega_2$ ,  $\phi_1^i$ , and  $\varphi^i = \phi_1^i - \phi_2^i$  are adjustable parameters of the classical fields. The nonstationary protected state  $|\Psi(t)\rangle$  is allowed for a nonadiabatic coherent evolution, which can be manipulated through those parameters  $\Omega_1$ ,  $\Omega_2$ ,  $\phi_1^i$  and  $\varphi^i$ . Under the assumption that  $\Omega_1 = N\Omega_2$  (where  $N$  is an integer and  $N \gg 1$ ), the evolution is periodic and the period is equal to  $\pi/\Omega_2$ . That is to say, we can obtain the protected initial state at time  $n\pi/\Omega_2$  ( $n = 1, 2, \dots$ ) and the concurrence of the system is invariable in the evolution. Concretely, suppose the initial state be  $|\Psi(0)\rangle = c_1(0)|e\rangle_A|e\rangle_B + c_2(0)|e\rangle_A|g\rangle_B + c_3(0)|g\rangle_A|e\rangle_B + c_4(0)|g\rangle_A|g\rangle_B$ , where  $c_i(0)$  are known complex constants and  $\sum_i |c_i(0)|^2 = 1$  ( $i = 1, 2, 3, 4$ ). Let  $t = 0$  in Eq. (10), we can obtain 4 equations, and then we can obtain the required parameters of the engineering reservoir  $\phi_1^i$  and  $\varphi^i$  and the coefficients  $\mu$  and  $r$  of the corresponding state  $|\Psi_r\rangle$  in the DFS, i.e., we can choose these parameters  $\phi_1^i$  and  $\varphi^i$  to project the given initial state  $|\Psi(0)\rangle$  to the corresponding state  $|\Psi_r\rangle$  in the DFS. As a simple example, we suppose that the initial state is a maximal entangled state  $|\Psi(0)\rangle$  (Bell states). From Eqs. (9) and (10), we obtain the parameters

**Table 1.** parameters of the engineering reservoir and the corresponding state  $|\Psi_r\rangle$ .

initial state $ \Psi(0)\rangle$	parameters of the engineering reservoir	corresponding state $ \Psi_r\rangle$ in DFS
$\frac{1}{\sqrt{2}}( e\rangle_A e\rangle_B +  g\rangle_A g\rangle_B)$	$\phi_1^A = \frac{\pi}{2}, \phi_1^B = \frac{\pi}{2}, \varphi^A = 0, \varphi^B = \pi$	$\frac{1}{\sqrt{2}}( -\rangle_A +\rangle_B -  +\rangle_A -\rangle_B)$
$\frac{1}{\sqrt{2}}( e\rangle_A e\rangle_B -  g\rangle_A g\rangle_B)$	$\phi_1^A = 0, \phi_1^B = 0, \varphi^A = 0, \varphi^B = \pi$	$\frac{i}{\sqrt{2}}( -\rangle_A +\rangle_B -  +\rangle_A -\rangle_B)$
$\frac{1}{\sqrt{2}}( e\rangle_A g\rangle_B +  g\rangle_A e\rangle_B)$	$\phi_1^A = -\frac{\pi}{2}, \phi_1^B = 0, \varphi^A = \pi, \varphi^B = \pi$	$\frac{-i}{\sqrt{2}}( -\rangle_A +\rangle_B -  +\rangle_A -\rangle_B)$
$\frac{1}{\sqrt{2}}( e\rangle_A g\rangle_B -  g\rangle_A e\rangle_B)$	$\phi_1^A = -\frac{\pi}{2}, \phi_1^B = -\frac{\pi}{2}, \varphi^A = \pi, \varphi^B = \pi$	$\frac{-1}{\sqrt{2}}( -\rangle_A +\rangle_B -  +\rangle_A -\rangle_B)$

of the engineering reservoir and the corresponding protected state  $|\Psi_r\rangle$  in the DFS (in table 1). Now we consider a more general example,  $|\Psi(0)\rangle = m|e\rangle_A|e\rangle_B + ne^{i\theta}|g\rangle_A|g\rangle_B$ , where  $m, n \geq 0$  satisfying  $m^2 + n^2 = 1$ , and  $\theta \in (0, 2\pi]$ . From Eqs. (9) and (10), we can obtain the parameters of the classical fields

$$\left. \begin{aligned} \phi_1^A &= \phi_1^B = \frac{1}{2}(\pi - \theta), \\ \varphi^A &= \frac{\pi}{2}[1 + \text{sign}(n - m)] - \arctan\left(\frac{2\sqrt{mn}}{|m-n|}\right), \\ \varphi^B &= \frac{\pi}{2}[1 + \text{sign}(n - m)] + \arctan\left(\frac{2\sqrt{mn}}{|m-n|}\right), \end{aligned} \right\} (m \neq n) \quad (11)$$

and the corresponding state  $|\Psi_r\rangle = \sqrt{mn} e^{i\frac{\theta}{2}}(|-\rangle_A|+\rangle_B - |+\rangle_A|-\rangle_B) + \sqrt{1-2mn} |+\rangle_A|+\rangle_B$  in the DFS. It must be noted that although many methods have been proposed to protect the entanglement from dissipation, our approach is different. Most of the schemes to protect entanglement is static, but ours is dynamic. More specifically the protected state goes through a cyclic nonadiabatic coherent evolution, but the degree of entanglement does not evolve and remains the same. In this way we can change one entangled state into another entangled state with the same degree of entanglement against dissipation.

All our discussions were based on Eq. (7), where spontaneous emission effects were neglected. However, spontaneous emission is the fundamental limiting factor for the existence of entanglement in a system of ions. The Eq. (6) describes the ionic system including the effect of spontaneous emission which introduces a coupling between the symmetric and antisymmetric subspaces. Next, we will analyze the spontaneous emission effects in our protected schemes. As an example, we consider that the two TL ions are initially prepared in

$$|\Psi_E\rangle = \frac{1}{\sqrt{2}}|g\rangle_A|g\rangle_B + \frac{1}{2}(|e\rangle_A|g\rangle_B - |g\rangle_A|e\rangle_B), \quad (12)$$

Without spontaneous decay, from Eqs. (9) and (10), we can see that by adjusting the parameters of the classical fields

$$\varphi^A = \varphi^B = \pi, \phi_1^A = 0, \phi_1^B = \pi, \text{ (or } \phi_1^A = \pi, \phi_1^B = 0), \quad (13)$$

the initial state  $|\Psi_E\rangle$  will undergo a coherent local evolution and the  $|\Psi_E(t)\rangle$  can be written as

$$\begin{aligned} |\Psi_E(t)\rangle = & \left( \frac{e^{-2i\Omega_2 t} \sin[\Omega_1 t]^2}{\sqrt{2}} - \frac{i \sin[2\Omega_1 t]}{2} \right) |e\rangle_A |e\rangle_B \\ & + \left( \frac{\cos[2\Omega_1 t]}{2} + \frac{i e^{-2i\Omega_2 t} \sin[2\Omega_1 t]}{2\sqrt{2}} \right) |e\rangle_A |g\rangle_B \\ & - \left( \frac{\cos[2\Omega_1 t]}{2} + \frac{i e^{-2i\Omega_2 t} \sin[2\Omega_1 t]}{2\sqrt{2}} \right) |g\rangle_A |e\rangle_B \\ & + \left( \frac{e^{-2i\Omega_2 t} \cos[\Omega_1 t]^2}{\sqrt{2}} + \frac{i \sin[2\Omega_1 t]}{2} \right) |g\rangle_A |g\rangle_B. \end{aligned} \quad (14)$$

Now instead of Eq.(7) we numerically solve Eq.(6), which include the effect of spontaneous emission for the given initial state  $|\Psi_E\rangle$ . We can compute the fidelity  $F = \text{Tr}[|\Psi_E(t)\rangle\langle\Psi_E(t)|R(t)\rho'_E(t)R^\dagger(t)]$ , where  $\rho'_E(t)$  is the solution of Eq.(6). Within the regime  $\Omega_1=10\Omega_2=100g$ ,  $g=500\gamma^A$ ,  $\gamma^A=\gamma^B$  and  $\kappa=3g$ , the fidelity  $F$  as a function of time is shown in Fig. 1. From Fig. 1, we can see that the spontaneous emissions move the system away from the protected state  $|\Psi_E(t)\rangle$ . If the spontaneous emission rate is very small, the fidelity reduces slowly, for example, when  $t=100\pi/\Omega_2$ , the fidelity is around 96.9%. Therefore, as long as the spontaneous emission rate  $\gamma_i$  is much smaller than the other relevant frequencies of the problem, we can neglect the spontaneous emission effect within a finite time, such as  $t \ll 100\pi/\Omega_2$  in Fig.1. It is worth stressing that our protected scheme might be realized experimentally. The setup of two atoms equally coupled to a cavity mode with possibility of individual addressing has already been demonstrated in [21]. The large cooperativity parameter ( $\Gamma = g^2/\kappa \gg \gamma$ ) has been obtained in a variety of recent experiments [22, 23, 24].

### 3. Relation between entanglement and geometric phase of the system

From above discussion, we can see that the protected two-ion state (9) is 'dynamic' rather than 'static'. If  $\Omega_1=N\Omega_2$  (where  $N$  is an integer and  $N \gg 1$ ), the system will undergo a cyclic coherent evolution. After a cyclic evolution, the system returns to its original state but may acquire a geometric phase. To compute the geometric phase we use the definition given in Ref. [25]. From Eq. (9), after a cyclic evolution,  $\tau = \pi/\Omega_2$ , the acquired geometric phase is

$$\beta^G(\tau) = i \int_0^\tau \langle \Psi(t) | \frac{d}{dt} | \Psi(t) \rangle dt = 2\pi(1 - r). \quad (15)$$

From Eq. (15) it can be seen that, the geometric phase of the whole system is only a simple linear function of the entanglement degree  $r$ , and have nothing to do with other system parameters. Next we calculate the geometric phase of the subsystems and to study the relation between the geometric phase of the subsystems and the entanglement degree. Generally speaking, the state of the subsystem is no longer a pure one, so we adopt the definition of geometric phase for mixed states under bilocal unitary evolution [26]. If the Schmidt coefficients are nondegenerate, after a cyclic evolution, the geometric

phase of the subsystem can be written as

$$\begin{aligned} \beta^i(\tau) = \arg[\sum_{k=1}^N p_k \langle \mu_k | U^i(\tau) | \mu_k \rangle \\ \times \exp(-\int_0^\tau \langle \mu_k | U^{i\dagger}(t) \dot{U}^i(t) | \mu_k \rangle dt)], \end{aligned} \quad (16)$$

where  $p_k$  is the Schmidt coefficient,  $|\mu_k\rangle$  is the corresponding eigenstate of the reduced density matrix  $\rho^i$  (obtained after tracing over the other ion) and  $U^i(t)$  is a local unitary evolution operator acting on the  $i$ -th ion.

For our system, if the initial state is not maximally entangled ( $r \neq 1$ ), the Schmidt coefficients are nondegenerate. Using Eq. (16), we can obtain the geometric phase of the subsystems under bilocal unitary evolution  $R(t)$

$$\beta^A = \beta^B = \arg[\cos(\sqrt{\frac{1-r}{1+r}}\pi) + i\sqrt{1-r^2}\sin(\sqrt{\frac{1-r}{1+r}}\pi)] \quad (17)$$

When the initial state is a maximal entangled state, i.e.,  $r = 1$ , the Schmidt coefficients are degenerate and the reduced density matrix of the subsystem at  $t = 0$  is

$$\rho^{A(B)}(0) = \frac{1}{2}I.$$

Because the system subjects to the bilocal unitary evolution  $R(t)$ , the reduced density matrix of the subsystem at any time  $t$  is

$$\begin{aligned} \rho^{A(B)}(t) &= \text{Tr}_{B(A)}[R^A(t) \otimes R^B(t) \rho(0) R^{A\dagger}(t) \otimes R^{B\dagger}(t)] \\ &= \frac{1}{2}I, \end{aligned} \quad (18)$$

which means

$$\beta^{A(B)} = 0. \quad (19)$$

From Eqs. (17) and (19), we can obtain the relation between the geometric phase of the subsystems and the entanglement degree. Again the geometric phase of the subsystems  $\beta^{A(B)}$  is also only a function of the entanglement degree  $r$  which can be seen from Eq. (17) and is shown in Fig. 2. It can be seen from Fig. 2 that  $\beta^{A(B)}$  is a monotonic decreasing function of  $r$ . If we could measure the geometric phase of the subsystems  $\beta^{A(B)}$ , we can infer the entanglement degree  $r$  of the protected entangled state.

#### 4. Conclusion

In this paper, we have considered two TL ions in a heavily damped cavity. Using engineering reservoir scheme, we have obtained a time-independent master equation, and then have found a DFS for this master equation, which can be nonadiabatically controlled by the system-reservoir parameters. We have achieved a class of decoherence-free cyclic evolution of the entangled state. Finally, we have calculated the geometric phases of the whole system and the subsystems under the nonadiabatic coherent evolution, and have found that there is one-to-one correspondence between the geometric phase of the whole system, the geometric phase of subsystems and the entanglement degree  $r$ .



## Acknowledgment

This work was supported by National Natural Science Foundation of China (Grants No. 10974016, No. 11005008, and No. 11075013).

## References

- [1] Shor P W 1995 *Phys. Rev. A* **52** R2493
- [2] Steane A M 1996 *Phys. Rev. Lett.* **77** 793
- [3] Ekert A and Macchiavello C 1996 *Phys. Rev. Lett.* **77** 2585
- [4] Zanardi P and Rasetti M 1997 *Phys. Rev. Lett.* **79** 3306
- [5] Duan L M and Guo G C 1998 *Phys. Rev. A* **57** 737
- [6] Lidar D A, Chuang I L and Whaley K B 1998 *Phys. Rev. Lett.* **81** 2594
- [7] Bacon D, Kempe J, Lidar D A and Whaley K B 2000 *Phys. Rev. Lett.* **85** 1758
- [8] Agarwal G S 1999 *Phys. Rev. A* **61** 013809
- [9] Viola L, Knill E and Lloyd S 1999 *Phys. Rev. Lett.* **82** 2417
- [10] Vitali D and Tombesi P 1999 *Phys. Rev. A* **59** 4178
- [11] Poyatos J F, Cirac J I and Zoller P 1996 *Phys. Rev. Lett.* **77** 4728
- [12] Mohseni M, Lundeen J S, Resch K J and Steinberg A M 2003 *Phys. Rev. Lett.* **91** 187903
- [13] Bourennane M, Eibl M, Gaertner S, Kurtsiefer C, Cabello A and Weinfurter H 2004 *Phys. Rev. Lett.* **92** 107901
- [14] Langer C, Ozeri R, Jost J D, Chiaverini J, DeMarco B, Ben-Kish A, Blakestad R B, Britton J, Hume D B, Itano W M, Leibfried D, Reichle R, Rosenband T, Schaetz T, Schmidt P O and Wineland D J 2005 *Phys. Rev. Lett.* **95** 060502
- [15] Carvalho A R R, Milman P, de Matos Filho R L and Davidovich L 2001 *Phys. Rev. Lett.* **86** 4988
- [16] Prado F O, Duzzioni E I, Moussa M H Y, de Almeida N G and Villas-Bôas C J 2009 *Phys. Rev. Lett.* **102** 073008
- [17] Carollo A, Paternostro G M, Lozinski A, Santos M F, and Vedral V 2006 *Phys. Rev. Lett.* **96** 150403
- [18] Wang J, Wiseman H M and Milburn G J 2005 *Phys. Rev. A* **71** 042309
- [19] Carvalho A R R, Reid A J S and Hope J J 2008 *Phys. Rev. A* **78** 012334
- [20] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
- [21] Nußmann S, Hijlkema M, Weber B, Rohde F, Rempe G and Kuhn A 2005 *Phys. Rev. Lett.* **95** 173602
- [22] Hood C J, Kimble H J and Ye J 2001 *Phys. Rev. A* **64** 033804
- [23] Boozer A D, Boca A, Miller R, Northup T E and Kimble H J 2006 *Phys. Rev. Lett.* **97** 083602
- [24] Maunz P, Puppe T, Schuster I, Syassen N, Pinkse P W H and Rempe G 2005 *Phys. Rev. Lett.* **94** 033002
- [25] Mukunda N and Simon R 1993 *Ann. Phys. (Leipzig)* **228** 205
- [26] Tong D M, Sjöqvist E, Kwek L C, Oh C H and Ericsson M 2003 *Phys. Rev. A* **68** 022106

## CAPTIONS

Fig.1 The fidelity  $F$  as a function of time.

Fig.2 The geometric phase of the subsystems  $\beta^{A(B)}$  as a function of entanglement degree  $r$ .



